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Asymptotic Analysis of Darcy-Brinkman-Boussinesq Model for Convection in Porous Media

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Abstract

In this article, we study the asymptotic behavior of the infinite Prandtl-Darcy number Darcy-Brinkman-Boussinesq system. We derive the asymptotic expansion with respect to the Brinkman-Darcy number, which improves the result obtained by Kelliher, et al. [1].

Keywords: Boundary layers, Infinite Prandtl-Darcy number, Darcy-Brinkman-Boussinesq system, Brinkman-Darcy number, Asymptotic analysis, Uniform convergence.
MSC (2000): 76D09, 76D10, 35Q35, 35B25, 76M45

Contents

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1. Introduction

Convection phenomena in porous media are relevant to a variety of science and engineering problems ranging from geothermal energy transport to fiberglass insulation [2]. The purpose of this paper is to investigate the approximations of the infinite Prandtl-Darcy number Darcy-Brinkman-Boussinesq system as the Prandtl-Darcy number goes to zero.

Here, we consider the following infinite Prandtl-Darcy number Darcy-Brinkman-Boussinesq system (IPDDBB),

$$\begin{cases} -\varepsilon^2 \Delta \vec{u}^\varepsilon + \vec{u}^\varepsilon + \nabla p^\varepsilon = \gamma T^\varepsilon \vec{k}, & \text{in } [0,T]\times\Omega, \\ div \vec{u}^\varepsilon = 0, & \text{in } [0,T]\times\Omega, \\ \partial_t T^\varepsilon + \vec{u}^\varepsilon \cdot \nabla T^\varepsilon = \Delta T^\varepsilon, & \text{in } [0,T]\times\Omega, \\ \vec{u}^\varepsilon = \vec{u}_0, T^\varepsilon = T_0, & \text{at } t=0, \\ \vec{u}^\varepsilon |_{z=0,1} = 0, T^\varepsilon |_{z=0} = 1, T^\varepsilon |_{z=1} = 0. \end{cases} \tag{1.1}$$

Where ε^2 is the Brinkman-Darcy number, γ is the Rayleigh-Darcy number, $\vec{k}$ is the unit normal vector directed upward (the positive z direction) and $\Omega = (0,2\pi)^2 \times (0,1)$ is a 3-dimensional channel, periodic in the $x-$ and $y-$ directions.

Formally setting the Brinkman-Darcy number to zero, we arrive at the following infinite Prandtl-Darcy number Darcy-Boussinesq system (IPDDB),

$$\begin{cases} \vec{u}^0 + \nabla p^0 = \gamma T^0 \vec{k}, & \text{in } [0,T]\times\Omega, \\ div \vec{u}^0 = 0, & \text{in}[0,\mathrm{T}]\times\Omega, \\ \partial_t T^0 + \vec{u}^0 \cdot \nabla T^0 = \Delta T^0, & \text{in}[0,\mathrm{T}]\times\Omega \, . \\ \vec{u}^0 = \vec{u}_0, T^0 = T_0, & \text{at } t=0, \\ u_3^0 |_{z=0,1} = 0, T^0 |_{z=0} = 1, T^0 |_{z=1} = 0. \end{cases} \tag{1.2}$$

The well-posedness and further regularity of (1.1) and (1.2) was established in [1, 3, 4]. Payne and Straughan [5] have established the convergence in L^2 on any finite time interval of the solutions of the IPDDBB to those of the IPDDB without resolving the boundary layer. However, we can not expect a convergence result of $\vec{u}^\varepsilon$ to $\vec{u}^0$ in the uniform space since they do no have the same traces on the boundary. This question was addressed in Ref. [1], in which the authors gave a representation of the the DBB solutions $\vec{u}^\varepsilon$ to the boundary and proved convergence results in several Sobolev spaces, especially, the uniform convergence in the case $\Omega \in \mathbb{R}^2$.

There is an abundant literature on boundary layer associated with incompressible flows and the related question of vanishing viscosity (see for instance [6-13] among many others).

Our interest is to derive the complete asymptotic expansion for $\vec{u}^\varepsilon$, when ε goes to zero. This is similar to the case of the boundary layer for the incompressible Navier-Stokes equations flows. We borrow from the work on Navier-Stokes and related systems, especially ideas and techniques in terms of corrector, weighted estimates, differential treatment of the tangential direction(s) and anisotropic embedding.

The article is organized as follows. Sect. 2 deals with the boundary layers. In Sect. 3, we show how to choose and construct the correctors of all orders, and propose our main results. Sect. 4 is devoted to the convergence in energy (L^2) space. Finally, in Sect. 5, we complete the proof of the convergence theorem in uniform norm.

2. Some Results of the Boundary Layers

Throughout this article, without particular mark, the constants, for example C, are irrelevant to ε. Since the boundary layers appear near the boundary at z direction, the $x-$ and $y-$ directions are denoted by $\vec{\tau}$.

In order to study the boundary layers, we introduce some function sets X^m, X_0^m and X_1^m, where $m \in \mathbb{N}$.

Definition 2.1 We say a function $\theta^\varepsilon \in X^m$, if and only if $\theta^\varepsilon \in H^m((0,T)\times\Omega)$ and the following inequalities hold

$$\| z^s (1-z)^s \partial_t^\alpha \partial_\tau^\beta \partial_z^\gamma \theta^\varepsilon \|_{L^2((0,T)\times\Omega)} \le C\varepsilon^{s-|\gamma|+\frac{1}{2}}, \tag{2.1}$$

$$\| z^s (1-z)^s \partial_t^\alpha \partial_\tau^\beta \partial_z^\gamma \theta^\varepsilon dz \|_{L^2_{t,\tau}(L^1_z)} \le C\varepsilon^{s+1-|\gamma|}, \tag{2.2}$$

where $s \ge 0$ and multi-index α, β, γ satisfying $|\alpha| + |\beta| + |\gamma| \le m$.

Definition 2.2 We say a function $\theta^\varepsilon \in X_0^m$, if and only if $\theta^\varepsilon \in H^m((0,T)\times\Omega)$ and the following inequalities hold

$$\| z^s \partial_t^\alpha \partial_\tau^\beta \partial_z^\gamma \theta^\varepsilon \|_{L^2((0,T)\times\Omega)} \le C\varepsilon^{s-|\gamma|+\frac{1}{2}}, \tag{2.3}$$

$$\| z^s \partial_t^\alpha \partial_\tau^\beta \partial_z^\gamma \theta^\varepsilon dz \|_{L^2_{t,\tau}(L^1_z)} \le C\varepsilon^{s+1-|\gamma|}, \tag{2.4}$$

where $s \ge 0$ and multi-index α, β, γ satisfying $|\alpha| + |\beta| + |\gamma| \le m$.

Definition 2.3 We say a function $\theta^\varepsilon \in X_1^m$, if and only if $\theta^\varepsilon \in H^m((0,T)\times\Omega)$ and the following inequalities hold

$$\| (1-z)^s \partial_t^\alpha \partial_\tau^\beta \partial_z^\gamma \theta^\varepsilon \|_{L^2((0,T)\times\Omega)} \le C\varepsilon^{s-|\gamma|+\frac{1}{2}}, \tag{2.5}$$

$$\|(1-z)^s \partial_t^\alpha \partial_\tau^\beta \partial_z^\gamma \theta^\varepsilon dz\|_{L^2_{t,\tau}(L^1_z)} \le C\varepsilon^{s+1-|\gamma|}, \quad (2.6)$$

where $s \ge 0$ and multi-index α, β, γ satisfying $|\alpha|+|\beta|+|\gamma| \le m$.

Definition 2.4 We say a function $\theta^\varepsilon \in Y^m$ if and only if $\|\theta^\varepsilon\|_{H^m} \le C$.

For the following results, we can refer to Xie and Li [12] and Temam [14].

Lemma 2.1 Let $\vec{f}^\varepsilon \in X^{m+2}$ and $\psi \in Y^{m+1}((0,T)\times\partial\Omega)$. Then there exist $\vec{\theta}^j$ $(j=1,2,3,4)$, s.t. $\vec{\theta}^\varepsilon = \vec{\theta}^1 + \varepsilon\vec{\theta}^2, \vec{\theta}^1 \in X^m, \vec{\theta}^2 \in Y^m, \vec{\theta}^3 \in X^m, \vec{\theta}^4 \in Y^m$, satisfying

$$\begin{cases} -\varepsilon^2 \dfrac{\partial^2}{\partial z^2}\vec{\theta}^\varepsilon + \vec{\theta}^\varepsilon + \nabla q^\varepsilon = f^\varepsilon + \varepsilon\vec{\theta}^3 + \varepsilon\vec{\theta}^4, & \text{in } \Omega, \\ div\vec{\theta}^\varepsilon = 0, & \text{in } \Omega, \\ \vec{\theta}^\varepsilon = (\psi_1, \psi_2, 0) & \text{on } \partial\Omega. \end{cases} \quad (2.7)$$

Lemma 2.2 Suppose $\vec{f}$, $\vec{u}^0, g, T^0 \in Y^m$ and some compatibility conditions up to m hold. Then the solutions of the equations

$$\begin{cases} \vec{u} + \nabla p = \gamma T\vec{k} + \vec{f}, & \text{in } (0,t^*)\times\Omega, \\ \nabla\cdot\vec{u} = 0, & \text{in } (0,t^*)\times\Omega, \\ \partial_t T + \vec{u}^0\cdot\nabla T + \vec{u}\cdot\nabla T^0 = \Delta T + g, & \text{in } (0,t^*)\times\Omega, \\ \vec{u}\cdot\vec{n} = 0, T = 0, & \text{on } \partial\Omega, \\ T = 0, & \text{at } t = 0. \end{cases} \quad (2.8)$$

satisfy $\vec{u} \in Y^m, T \in Y^{m+1}$.

3. Derivation of the Asymptotic Expansion Equations and Main Results

Our interest here is in the asymptotic behavior of the solutions of the IPDDBB equations (1.1) at the small Brinkman-Darcy number.

Considering the physical boundary, we propose a sequence of approximations

$$\vec{u}^\varepsilon = \vec{W}_u^{k,\varepsilon} + \sum_{j=0}^{k}\varepsilon^j(\vec{u}^j + \vec{\theta}^j), T^\varepsilon = W_T^{k,\varepsilon} + \sum_{j=0}^{k}\varepsilon^j(T^j + \vartheta^j). \quad (3.1)$$

Now, taking (3.1) into IPDDBB equations (1.1), we arrange the terms in the following order

$$\vec{u}^0 - \gamma T^0\vec{k} - \varepsilon^2\Delta\vec{u}^0 \quad (3.2)$$

$$-\varepsilon^2\frac{\partial^2}{\partial z^2}\vec{\theta}^0 + \vec{\theta}^0 - \gamma\vartheta^0\vec{k} - \varepsilon^2\Delta_\tau\vec{\theta}^0 \quad (3.3)$$

$$\vdots$$

$$+\varepsilon^k\vec{u}^k - \varepsilon^k\gamma T^k\vec{k} - \varepsilon^{k+2}\Delta\vec{u}^k \quad (3.4)$$

$$-\varepsilon^{k+2}\frac{\partial^2}{\partial z^2}\vec{\theta}^k + \varepsilon^k\vec{\theta}^k - \varepsilon^k\gamma\vartheta^k\vec{k} - \varepsilon^{k+2}\Delta_\tau\vec{\theta}^k \quad (3.5)$$

$$-\varepsilon^2\Box\vec{W}_u^{k,\varepsilon} + \vec{W}_u^{k,\varepsilon} + \nabla p^\varepsilon - \gamma W_T^{k,\varepsilon}\vec{k} = 0 \quad (3.6)$$

$$\frac{\partial}{\partial t}T^0 + \vec{u}^0\cdot\nabla T^0 - \Delta T^0 + \vec{\theta}^0\cdot\nabla T^0 \quad (3.7)$$

$$-\frac{\partial^2}{\partial z^2}\vartheta^0 + \frac{\partial}{\partial t}\vartheta^0 + \vec{u}^0\cdot\nabla\vartheta^0 + \vec{\theta}^0\cdot\nabla\vartheta^0 - \Delta_\tau\vartheta^0 \quad (3.8)$$

$$\vdots$$

$$\begin{aligned} &+\varepsilon^k\frac{\partial}{\partial t}T^k + \varepsilon^k\vec{u}^0\cdot\nabla T^k + \varepsilon^k\vec{u}^k\cdot\nabla T^0 - \varepsilon^k\Delta T^k + \sum_{j=1}^{k-1}\varepsilon^{k+j}\vec{u}^j\cdot\nabla T^k \\ &+\sum_{j=1}^{k}\varepsilon^{k+j}\vec{u}^k\cdot\nabla T^j + \sum_{j=0}^{k-1}\varepsilon^{k+j}\vec{\theta}^j\cdot\nabla T^k + \sum_{j=0}^{k}\varepsilon^{k+j}\vec{\theta}^k\cdot\nabla T^j \end{aligned} \quad (3.9)$$

$$\begin{aligned} &-\varepsilon^k\frac{\partial^2}{\partial z^2}\vartheta^k + \varepsilon^k\frac{\partial}{\partial t}\vartheta^k - \varepsilon^k\Delta_\tau\vartheta^k + \sum_{j=0}^{k-1}\varepsilon^{k+j}\vec{u}^j\cdot\nabla\vartheta^k \\ &+\sum_{j=0}^{k}\varepsilon^{k+j}\vec{u}^k\cdot\nabla\vartheta^j + \sum_{j=0}^{k-1}\varepsilon^{k+j}\vec{\theta}^j\cdot\nabla\vartheta^k + \sum_{j=0}^{k}\varepsilon^{k+j}\vec{\theta}^k\cdot\nabla\vartheta^j \end{aligned} \quad (3.10)$$

$$+\frac{\partial}{\partial t}W_T^{k,\varepsilon} + \vec{u}^\varepsilon\cdot\nabla W_T^{k,\varepsilon} + \vec{W}_u^{k,\varepsilon}\cdot\nabla T^\varepsilon - \vec{W}_u^{k,\varepsilon}\cdot\nabla W_T^{k,\varepsilon} - \Box W_T^{k,\varepsilon} = 0. \quad (3.11)$$

The IPDDB equations (1.2) can be constructed by (3.2) and (3.7). Thanks to the consistent conditions, $\vec{u}^0 \in Y^{4k+6}, T^0 \in Y^{4k+7}$ (the consistent conditions of IPDDB and the following linear IPDDB equations are shown in the end of this section).

From (3.3) and applying Lemma 2.1, we know there exist $\vec{\theta}^0 = \vec{\theta}^{0,1} + \varepsilon\vec{\theta}^{0,2}$, $\theta^{0,3} \in X^{4k+4}$, $\theta^{0,4} \in Y^{4k+4}$, which satisfy the follows:

$$\begin{cases} -\varepsilon^2 \dfrac{\partial^2}{\partial z^2}\vec{\theta}^0 + \vec{\theta}^0 + \nabla q^0 = \varepsilon\vec{\theta}^{0,3} + \varepsilon\vec{\theta}^{0,4}, & \text{in } \Omega, \\ div\vec{\theta}^0 = 0, & \text{in } \Omega, \\ \vec{\theta}^0 = -\vec{u}^0, & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

From (3.8), assume that ϑ^0 satisfies

$$\begin{cases} -\varepsilon^2 \dfrac{\partial^2}{\partial z^2}\vartheta^0 = 0 \text{ in } \Omega, \\ \vartheta^0 = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.13}$$

The remainder terms are

$$\tilde{f}^0 = -\varepsilon^2\square\vec{u}^0 - \gamma\vartheta^0\vec{k} - \varepsilon^2\square_\tau\theta^0 + \varepsilon\vec{\theta}^{0,3} + \varepsilon\vec{\theta}^{0,4} = \varepsilon\tilde{f}^{0,1} + \varepsilon\tilde{f}^{0,2}, \tag{3.14}$$

$$\tilde{g}^0 = \vec{\theta}^0 \cdot \nabla T^0 + \frac{\partial}{\partial t}\vartheta^0 + \vec{u}^0 \cdot \nabla\vartheta^0 + \vec{\theta}^0 \cdot \nabla\vartheta^0 - \square_\tau\vartheta^0 = \varepsilon\tilde{g}^{0,1} + \varepsilon\tilde{g}^{0,2}, \tag{3.15}$$

where $\tilde{f}^{0,1} \in X^{4k+2}$, $\tilde{f}^{0,2} \in Y^{4k+2}$, $\tilde{g}^{0,1} \in \frac{1}{\varepsilon}X^{4k+2}$ and $\tilde{g}^{0,2} \in Y^{4k+2}$.

$\tilde{f}^{0,2}$ and $\tilde{g}^{0,2}$ are used to construct the equations with respect to $\vec{u}^1$ and T^1, $\tilde{f}^{0,1}$ and $\tilde{g}^{0,1}$ are used to construct the equations with respect to $\vec{\theta}^1$ and ϑ^1, respectively. Step by step, all the equations are constructed.

Now, after obtaining $\tilde{f}^{k-1,1} \in X^6$, $\tilde{g}^{k-1,1} \in \frac{1}{\varepsilon}X^6$, $\tilde{f}^{k-1,2} \in Y^6$ and $\tilde{g}^{k-1,2} \in Y^6$, we give the equations with resect to $\vec{u}^k$, T^k, $\vec{\theta}^k$ and ϑ^k.

$$\begin{cases} \vec{u}^k + \nabla p^k = \gamma T^k\vec{k} + \tilde{f}^{k-1,2}, & \text{in } (0,\mathrm{T})\times\Omega, \\ \nabla\cdot\vec{u}^k = 0, & \text{in } (0,\mathrm{T})\times\Omega, \\ \dfrac{\partial}{\partial t}T^k + \vec{u}^0\cdot\nabla T^k + \vec{u}^k\cdot\nabla T^0 = \Delta T^k + \tilde{g}^{k-1,2}, & \text{in } (0,\mathrm{T})\times\Omega, \\ \vec{u}^k\cdot\vec{n} = 0, T^k = 0, & \text{on } \partial\Omega, \\ T^k = 0, & \text{at } t = 0. \end{cases} \tag{3.16}$$

$$\begin{cases} -\varepsilon^2 \dfrac{\partial^2}{\partial z^2}\vec{\theta}^k + \vec{\theta}^k + \nabla q^k = \tilde{f}^{k-1,1} + \varepsilon\vec{\theta}^{k,3} + \varepsilon\vec{\theta}^{k,4}, & \text{in } \Omega, \\ div\vec{\theta}^k = 0, & \text{in } \Omega, \\ \vec{\theta}^k = -\vec{u}^k, & \text{on } \partial\Omega. \end{cases} \tag{3.17}$$

$$\begin{cases} -\varepsilon^2 \dfrac{\partial^2}{\partial z^2}\vartheta^k = \tilde{g}^{k-1,1} \text{ in } \Omega \\ \vartheta^0 = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.18}$$

Here $\vec{u}^k \in Y^6$, $T^k \in Y^7$, $\vec{\theta}^k = \vec{\theta}^{k,1} + \varepsilon\vec{\theta}^{k,2}$, $\vec{\theta}^{k,1} \in X^4, \vec{\theta}^{k,2} \in Y^4$, $\vec{\theta}^{k,3} \in X^4$, $\vec{\theta}^{k,4} \in Y^4$, $\vartheta^k = \varepsilon\vartheta^{k,1} + \varepsilon\vartheta^{k,2}$, $\vartheta^{k,1} \in X^6$ and $\vartheta^{k,2} \in Y^6$.

The remainder terms are

$$\tilde{f}^k = -\varepsilon^2\Delta\vec{u}^k - \gamma\vartheta^k\vec{k} - \varepsilon^2\Delta_\tau\vec{\theta}^k + \varepsilon\vec{\theta}^{k,3} + \varepsilon\vec{\theta}^{k,4} = \varepsilon\tilde{f}^{k,1} + \varepsilon\tilde{f}^{k,2}, \tag{3.19}$$

$$\begin{aligned} \tilde{g}^k = & \sum_{j=1}^{k-1}\varepsilon^j\vec{u}^j\cdot\nabla T^k + \sum_{j=1}^{k}\varepsilon^j\vec{u}^k\cdot\nabla T^j + \sum_{j=0}^{k-1}\varepsilon^j\vec{\theta}^j\cdot\nabla T^k + \sum_{j=0}^{k}\varepsilon^j\vec{\theta}^k\cdot\nabla T^j \\ & + \varepsilon^k\frac{\partial}{\partial t}\vartheta^k - \varepsilon^k\Delta_\tau\vartheta^k + \sum_{j=0}^{k-1}\varepsilon^{k+j}\vec{u}^j\cdot\nabla\vartheta^k + \sum_{j=0}^{k}\varepsilon^{k+j}\vec{u}^k\cdot\nabla\vartheta^j \\ & + \sum_{j=0}^{k-1}\varepsilon^{k+j}\vec{\theta}^j\cdot\nabla\vartheta^k + \sum_{j=0}^{k}\varepsilon^{k+j}\vec{\theta}^k\cdot\nabla\vartheta^j \\ = & \varepsilon\tilde{f}^{k,1} + \varepsilon\tilde{f}^{k,2}, \end{aligned} \tag{3.20}$$

where $\tilde{f}^{k,1} \in X^2$, $\tilde{f}^{k,2} \in Y^2$, $\tilde{g}^{k,1} \in \frac{1}{\varepsilon} X^2$, $\tilde{g}^{k,2} \in Y^2$.

Thus, the error equations are

$$\begin{cases} -\varepsilon^2 \Delta \vec{W}_u^{k,\varepsilon} + \vec{W}_u^{k,\varepsilon} + \nabla p^{k,\varepsilon} = \gamma W_T^{k,\varepsilon} \vec{k} + \varepsilon^{k+1} \tilde{f}^{k,1} + \varepsilon^{k+1} \tilde{f}^{k,2}, \\ \nabla \cdot \vec{W}_u^{k,\varepsilon} = 0, \\ \frac{\partial}{\partial t} W_T^{k,\varepsilon} + \vec{u}^{\varepsilon} \cdot \nabla W_T^{k,\varepsilon} + \vec{W}_u^{k,\varepsilon} \cdot \nabla T^{\varepsilon} - \vec{W}_u^{k,\varepsilon} \cdot \nabla W_T^{k,\varepsilon} - \Delta W_T^{k,\varepsilon} = \varepsilon^{k+1} \tilde{g}^{k,1} + \varepsilon^{k+1} \tilde{g}^{k,2}, \\ \vec{W}_u^{k,\varepsilon} = 0, W_T^{k,\varepsilon} = 0 \text{ at z=0, z=1}, \\ W_T^{k,\varepsilon} = 0 \text{ at t=0}. \end{cases} \quad (3.21)$$

The results about the adjusted differences $\vec{W}_u^{0,\varepsilon}$ and $W_T^{0,\varepsilon}$ can be found in Kelliher, et al. [1]

Theorem 3.1 ([1]) Let $\vec{u}^0, T^0 \in C^k([0,T] \times \bar{\Omega}), k \geq 6$. Then

$$\begin{aligned} &\| \vec{W}_u^{0,\varepsilon} \|_{L^\infty(0,T;L^2)} \leq C\varepsilon, |\vec{W}_u^{0,\varepsilon}\|_{L^\infty(0,T;H^1)} \leq C, \\ &\| W_T^{0,\varepsilon} \|_{L^\infty(0,T;L^2)} \leq C\varepsilon, \| W_T^{0,\varepsilon} \|_{L^2(0,T;H^1)} \leq C\varepsilon. \end{aligned} \quad (3.22)$$

If $\Omega \in \mathbb{R}^2$,

$$\begin{aligned} &\| W_u^{0,\varepsilon} \|_{L^\infty(0,T;H^1)}, \| \frac{\partial}{\partial t} W_T^{0,\varepsilon} \|_{L^2(0,T;L^2)} \leq C\varepsilon^{\frac{1}{2}}, \\ &\| \vec{W}_u^{0,\varepsilon} \|_{L^\infty((0,T)\times\Omega)} \leq C\varepsilon^{\frac{1}{4}}, \\ &\| W_T^{0,\varepsilon} \|_{L^\infty((0,T)\times\Omega)} \leq C\varepsilon^{\frac{3}{4}}. \end{aligned} \quad (3.23)$$

Each of the constants, C, depends only on T_0 and T.

Now, we show the compatibility conditions of the above equations.

Lemma 3.1 For any $k \in \mathbb{N}$, we assume

$$\begin{aligned} &T_0 = T_0(z), T_0|_{z=0} = 1, T_0|_{z=1} = 0, \\ &\text{and } \frac{\partial^{2j}}{\partial z^{2j}} T_0 = 0 \text{ at } z = 0,1, j \in \mathbb{N}^+, j \leq 4k+6. \end{aligned} \quad (3.24)$$

Then, the above IPDDBB, IPDDB and linear IPDDB equations have the compatibility conditions.

Proof. From $T_0 = T_0(z)$, we know $P(T_0\vec{k}) = 0$, where P is the projection operator from $L^2(\Omega)$ to its divergence-free subspace according to Hodge decomposition.

Then by the first equation of IPDDBB system (1.1}) and IPDDB system (1.2), $\vec{u}^{\varepsilon}|_{t=0} = 0$, $\vec{u}^0|_{t=0} = 0$. Furthermore $\tilde{f}^0|_{t=0} = 0$, $\tilde{g}^0|_{t=0} = 0$. From the equations, we have $\frac{\partial}{\partial t} T^{\varepsilon}|_{t=0} = (\frac{\partial}{\partial t} T^0)|_{t=0} = \Delta T_0 = 0$ at $z = 0, z = 1$.

Let $\frac{\partial^j}{\partial t^j} T^{\varepsilon}|_{t=0} = \frac{\partial^j}{\partial t^j} T^0|_{t=0} = \Delta^j T_0$, $j \leq j_0$. Then $P(\frac{\partial^j}{\partial t^j} T_0 \vec{k}) = 0$, and by the first equation of IPDDBB system (1.1) and IPDDB system (1.2), $\frac{\partial^j}{\partial t^j} \vec{u}^{\varepsilon}|_{t=0} = 0$, $\frac{\partial^j}{\partial t^j} \vec{u}^0|_{t=0} = 0$. Furthermore $\frac{\partial^j}{\partial t^j} \tilde{f}^0|_{t=0} = 0$, $\frac{\partial^j}{\partial t^j} \tilde{g}^0|_{t=0} = 0$. From the equations, we have $(\frac{\partial^{j_0+1}}{\partial t^{j_0+1}} T^{\varepsilon})|_{t=0} = \Delta \frac{\partial^{j_0}}{\partial t^{j_0}} T^0 = \Delta^{j_0+1} T_0 = 0$. Then applying the conductive method and (3.24), IPDDBB system and IPDDB system have the compatibility conditions up to $4k+6$.

Noting that $\frac{\partial^j}{\partial t^j} \tilde{f}^0 = \frac{\partial^j}{\partial t^j} \tilde{g}^0 = 0$ at $t = 0$ and following the same way, the compatibility conditions of linear IPDDB system with respect to $\vec{u}^1$ and T^1 can be obtained. It is trivial to prove the result by the conductive method..

The previous regularity results of the IPDDB equations and linear IPDDB equations hold.

Corollary 3.1 Assume (3.24) hold. Then,

$$\frac{\partial}{\partial t} W_T^{k,\varepsilon} = 0, \frac{\partial^2}{\partial t^2} W_T^{k,\varepsilon} = 0 \text{ at } t = 0. \quad (3.25)$$

Our main results are as follows

Theorem 3.2 Suppose (3.24) hold. Then for any $s < \frac{1}{2}$,

$$\| \vec{W}_u^{k,\varepsilon} \|_{L^\infty(0,T;L^2)} \leq C\varepsilon^{k+1}, \| W_T^{k,\varepsilon} \|_{L^\infty(0,T;L^2)} \leq C\varepsilon^{k+1}, \quad (3.26)$$

$$\| \vec{W}_u^{k,\varepsilon} \|_{L^\infty(0,T;L^\infty)} \leq C\varepsilon^{k+s}, \| W_T^{k,\varepsilon} \|_{L^\infty(0,T;L^\infty)} \leq C\varepsilon^{k+1}, \quad (3.27)$$

where the constant, C, depends only on T_0 and T.

Remark 3.1 The proof is shown in the next two sections. The estimate (3.26) is simply obtained in Sect. 4 by energy estimates. Nevertheless, for the uniform estimate (3.27) some technical results are needed. More precisely, we will use an annpropriate anisotropic embedding theorem in the proof of the estimate (3.27) which is the subject of Sect. 5.

4. Proof of the Convergence in the Energy Norm

Since we have obtained k-order boundary layers, it is trivial to derive the L^2 estimates of the adjusted difference by energy estimates.

Multiplying the first equation of Eqs. (3.21) by $\vec{W}_u^{k,\varepsilon}$ and integrating in Ω, we have

$$\varepsilon^2\|\nabla\vec{W}_u^{k,\varepsilon}\|_{L^2}^2+\|\vec{W}_u^{k,\varepsilon}\|_{L^2}^2\le\frac{1}{2}\|\vec{W}_u^{k,\varepsilon}\|_{L^2}^2+C\|W_T^{k,\varepsilon}\|_{L^2}^2+C\varepsilon^{2k+2},\tag{4.1}$$

then,

$$\varepsilon^2\|\nabla\vec{W}_u^{k,\varepsilon}\|_{L^2}^2+\|\vec{W}_u^{k,\varepsilon}\|_{L^2}^2\le+C\|W_T^{k,\varepsilon}\|_{L^2}^2+C\varepsilon^{2k+2}.\tag{4.2}$$

Multiplying the third equation of Eqs.(3.21) by $W_T^{k,\varepsilon}$ and integrating in Ω, we have

$$\int_\Omega\frac{\partial}{\partial t}W_T^{k,\varepsilon}\cdot W_T^{k,\varepsilon}=\frac{1}{2}\frac{d}{dt}\|W_T^{k,\varepsilon}\|_{L^2}^2,\tag{4.3}$$

$$\int_\Omega\vec{u}^\varepsilon\cdot\nabla W_T^{k,\varepsilon}\cdot W_T^{k,\varepsilon}=0,\tag{4.4}$$

$$\begin{aligned}|\int_\Omega\vec{W}_u^{k,\varepsilon}\cdot\nabla(T^j+\vartheta^j)\cdot W_T^{k,\varepsilon}|&\le\|\nabla(T^j+\vartheta^j)\|_{L^\infty}\|\vec{W}_u^{k,\varepsilon}\|_{L^2}\|W_T^{k,\varepsilon}\|_{L^2}\\&\le C\|\vec{W}_u^{k,\varepsilon}\|_{L^2}^2+C\|W_T^{k,\varepsilon}\|_{L^2}^2,\end{aligned}\tag{4.5}$$

$$\int_\Omega\vec{W}_u^{k,\varepsilon}\cdot\nabla W_T^{k,\varepsilon}\cdot W_T^{k,\varepsilon}=0,\tag{4.6}$$

$$-\int_\Omega\Delta W_T^{k,\varepsilon}\cdot W_T^{k,\varepsilon}=\|\nabla W_T^{k,\varepsilon}\|_{L^2}^2,\tag{4.7}$$

$$\begin{aligned}|\int_\Omega\varepsilon^{k+1}\tilde{g}^{k,1}\cdot W_T^{k,\varepsilon}|&\le\varepsilon^{k+1}\|z(1-z)\tilde{g}^{k,1}\|_{L^2}\|\frac{W_T^{k,\varepsilon}}{z(1-z)}\|_{L^2}\\&\le C\varepsilon^{k+\frac{3}{2}}\|\nabla W_T^{k,\varepsilon}\|_{L^2}\quad(\text{byHardy'sinequality})\\&\le\frac{1}{2}\|W_T^{k,\varepsilon}\|_{L^2}^2+C\varepsilon^{2k+3},\end{aligned}\tag{4.8}$$

$$|\int_\Omega\varepsilon^{k+1}\tilde{g}^{k,2}\cdot W_T^{k,\varepsilon}|\le\varepsilon^{k+1}\|\tilde{g}^{k,2}\|_{L^2}^2\|W_T^{k,\varepsilon}\|_{L^2}\le C\|W_T^{k,\varepsilon}\|_{L^2}^2+C\varepsilon^{2k+2}.\tag{4.9}$$

Combining (4.2)-(4.9), we have

$$\frac{d}{dt}\|W_T^{k,\varepsilon}\|_{L^2}^2+\|\nabla W_T^{k,\varepsilon}\|_{L^2}^2\le C\|W_T^{k,\varepsilon}\|_{L^2}^2+C\varepsilon^{2k+2}.\tag{4.10}$$

By applying the Gronwall inequality, we obtain

$$\|W_T^{k,\varepsilon}\|_{L^\infty(0,T;L^2)}\le C\varepsilon^{k+1}.\tag{4.11}$$

Recalling (4.2), we have

$$\|\vec{W}_u^{k,\varepsilon}\|_{L^\infty(0,T;L^2)}\le C\varepsilon^{k+1},\|\nabla\vec{W}_u^{k,\varepsilon}\|_{L^\infty(0,T;L^2)}\le C\varepsilon^k.\tag{4.12}$$

This concludes the proof of (3.26) in Theorem 3.2.

5. Proof of the Uniform Convergence

Our object in this section is to obtain the L^∞ estimate of the adjusted differences and complete the proof of Theorem 3.2. We use the technique tools in Xie and Zhang [15], more precisely, by applying an appropriate anisotropic embedding theorem. And the $L^\infty(0,T;L^2)$ estimates of the derivatives are needed. The estimates of derivatives to t and $\vec{\tau}$ are derived by energy estimates without worrying about boundary conditions. The modified Gronwall inequality are employed to deal with the nonlinear terms. The estimates of the 1-order derivative to z can be obtained by (4.10).

We postpone the proof of Theorem until the following Modified Gronwall inequality is drawn just as in Xie and Zhang [15].

Lemma 5.1. [15] Suppose $y(0)=0$, $y(t)\in C^1$ and

$$\frac{d}{dt}y^2(t)\le C(\varepsilon^{-2k}y^4+2y^2+\varepsilon^{2k+2})\text{ for }t\le T,\tag{5.1}$$

where ε is a small positive constant and the constant C is not dependent on ε.

Then we have

$$y^2(t)\le C\varepsilon^{2k+2},\text{ for }t\le T.\tag{5.2}$$

Recalling (4.10), we have

Lemma 5.2

$$\|\nabla W_T^{k,\varepsilon}\|_{L^2}^2 \le C\varepsilon^{k+1}\|\frac{\partial}{\partial t}W_T^{k,\varepsilon}\|_{L^2} + C\varepsilon^{2k+2}. \qquad (5.3)$$

In view of Lemma 5.2, in order to obtain the $L^\infty(0,T;L^2)$ estimate of $\nabla W_T^{k,\varepsilon}$, now we show the $L^\infty(0,T;L^2)$ estimate of $\frac{\partial}{\partial t}W_T^{k,\varepsilon}$.

Differentiating Eqs. (3.21) in time and denoting $I_1 = \frac{\partial}{\partial t}\vec{W}_u^{k,\varepsilon}$ and $J_1 = \frac{\partial}{\partial t}W_T^{k,\varepsilon}$, we have

$$\begin{cases} -\varepsilon^2\Delta I_1 + I_1 + \nabla\frac{\partial}{\partial t}p^{k,\varepsilon} = \gamma J_1\vec{k} + \varepsilon^{k+1}\frac{\partial}{\partial t}\tilde{f}^{k,1} + \varepsilon^{k+1}\frac{\partial}{\partial t}\tilde{f}^{k,2}, \\ \nabla\cdot I_1 = 0, \\ \frac{\partial}{\partial t}J_1 + \vec{u}^\varepsilon\cdot\nabla J_1 + \frac{\partial}{\partial t}\vec{u}^\varepsilon\cdot\nabla W_T^{k,\varepsilon} + \vec{W}_u^{k,\varepsilon}\cdot\nabla\frac{\partial}{\partial t}T^\varepsilon + I_1\cdot\nabla T^\varepsilon \\ -\vec{W}_u^{k,\varepsilon}\cdot\nabla J_1 - I_1\cdot\nabla W_T^{k,\varepsilon} - \Delta J_1 = \varepsilon^{k+1}\frac{\partial}{\partial t}\tilde{g}^{k,1} + \varepsilon^{k+1}\frac{\partial}{\partial t}\tilde{g}^{k,2}, \\ I_1 = 0, J_1 = 0 \text{ at z=0, z=1}, \\ J_1 = 0 \text{ at } t = 0. \end{cases} \qquad (5.4)$$

In a same manner, multiplying the first equation of (5.4) by I_1 and integrating in Ω, since it is linear, we have

$$\varepsilon^2\|\nabla I_1\|_{L^2}^2 + \|I_1\|_{L^2}^2 \le C\|J_1\|_{L^2}^2 + C\varepsilon^{2k+2}. \qquad (5.5)$$

Multiplying the third equation of Eqs. (5.4) by J_1 and integrating in Ω, we have

$$\int_\Omega \frac{\partial}{\partial t}J_1\cdot J_1 = \frac{1}{2}\frac{d}{dt}\|J_1\|_{L^2}^2, \qquad (5.6)$$

$$\int_\Omega \vec{u}^\varepsilon\cdot\nabla J_1\cdot J_1 = 0, \qquad (5.7)$$

$$\begin{aligned} &|\int_\Omega \frac{\partial}{\partial t}(\vec{u}^j + \vec{\theta}^j)\cdot\nabla W_T^{k,\varepsilon}\cdot J_1| \\ &\le (\|\frac{\partial}{\partial t}\vec{u}^j\|_{L^\infty} + \|\frac{\partial}{\partial t}\vec{\theta}^j\|_{L^\infty})\|\nabla W_T^{k,\varepsilon}\|_{L^2}\|J_1\|_{L^2} \text{ (by Lemma 5.2)} \\ &\le C\|J_1\|_{L^2}^2 + C\varepsilon^{2k+2}, \end{aligned} \qquad (5.8)$$

$$\begin{aligned} |\int_\Omega \vec{W}_u^{k,\varepsilon}\cdot\nabla\frac{\partial}{\partial t}(T^j + \vartheta^j)\cdot J_1| &\le (\|\frac{\partial}{\partial t}T^j\|_{L^\infty} + \|\frac{\partial}{\partial t}\vartheta^j\|_{L^\infty})\|\vec{W}_u^{k,\varepsilon}\|_{L^2}\|J_1\|_{L^2} \\ &\le C\|J_1\|_{L^2}^2 + C\varepsilon^{2k+2}, \end{aligned} \qquad (5.9)$$

$$\begin{aligned} |\int_\Omega I_1\cdot\nabla(T^j + \vartheta^j)\cdot J_1| &\le (\|T^j\|_{L^\infty} + \|\vartheta^j\|_{L^\infty})\|I_1\|_{L^2}\|J_1\|_{L^2} \\ &\le C\|I_1\|_{L^2}^2 + C\|J_1\|_{L^2}^2, \end{aligned} \qquad (5.10)$$

$$\int_\Omega \vec{W}_u^{k,\varepsilon}\cdot\nabla J_1\cdot J_1 = 0, \qquad (5.11)$$

$$\begin{aligned} |\int_\Omega I_1\cdot\nabla W_T^{k,\varepsilon}\cdot J_1| &= |\int_\Omega I_1\cdot W_T^{k,\varepsilon}\cdot\nabla J_1| \\ &\le \|I_1\|_{L^4}\|W_T^{k,\varepsilon}\|_{L^4}\|\nabla J_1\|_{L^2} \text{ (by Gagliardo-Nirenberg inequality)} \\ &\le (\|I_1\|_{L^2} + \|\nabla I_1\|_{L^2})(\|W_T^{k,\varepsilon}\|_{L^2} + \|\nabla W_T^{k,\varepsilon}\|_{L^2})\|\nabla J_1\|_{L^2} \\ &\le \frac{1}{8}\|\nabla J_1\|_{L^2}^2 + C(\|I_1\|_{L^2}^2 + \|\nabla I_1\|_{L^2}^2)(\|W_T^{k,\varepsilon}\|_{L^2}^2 + \|\nabla W_T^{k,\varepsilon}\|_{L^2}^2) \\ &\text{(by (5.5) and Lemma 5.2)} \\ &\le \frac{1}{8}\|\nabla J_1\|_{L^2}^2 + C\varepsilon^{-2}\|J_1\|_{L^2}^2(\varepsilon^{k+1}\|J_1\|_{L^2} + \varepsilon^{2k+2}) \\ &\le \frac{1}{8}\|\nabla J_1\|_{L^2}^2 + C\|J_1\|_{L^2}^2 + C\varepsilon^{-2k}\|J_1\|_{L^2}^4 + C\varepsilon^{2k+2}, \text{ for k} \ge 1, \end{aligned} \qquad (5.12)$$

$$-\int_\Omega \Delta J_1\cdot J_1 = \|\nabla J_1\|_{L^2}^2, \qquad (5.13)$$

$$|\varepsilon^{k+1}\int_\Omega(\frac{\partial}{\partial t}\tilde{g}^{k,1}+\frac{\partial}{\partial t}\tilde{g}^{k,1})\cdot J_1|$$

$$\leq \varepsilon^{k+1}\| z(1-z)\frac{\partial}{\partial t}\tilde{g}^{k,1}\|_{L^2}\|\frac{J_1}{z(1-z)}\|_{L^2}+\varepsilon^{k+1}\|\frac{\partial}{\partial t}\tilde{g}^{k,2}\|_{L^2}\| J_1\|_{L^2} \quad (5.14)$$

$$\leq \frac{1}{8}\|\nabla J_1\|_{L^2}^2+C\| J_1\|_{L^2}^2+C\varepsilon^{2k+2},$$

Combining (5.5)-(5.14),

$$\frac{d}{dt}\| J_1\|_{L^2}^2+\|\nabla J_1\|_{L^2}^2\leq C(\varepsilon^{-2k}\| J_1\|_{L^2}^4+2\| J_1\|_{L^2}^4+\varepsilon^{2k+2}). \quad (5.15)$$

In view of (5.5), Lemma 5.1 and Lemma 5.2, we have

Lemma 5.3

$$\|\frac{\partial}{\partial t}W_T^{k,\varepsilon}\|_{L^\infty(0,T;L^2)}\leq C\varepsilon^{k+1},\|\nabla\frac{\partial}{\partial t}W_T^{k,\varepsilon}\|_{L^2(0,T;L^2)}\leq C\varepsilon^{k+1}, \quad (5.16)$$

$$\|\frac{\partial}{\partial t}\vec{W}_u^{k,\varepsilon}\|_{L^\infty(0,T;L^2)}\leq C\varepsilon^{k+1},\|\nabla\frac{\partial}{\partial t}\vec{W}_u^{k,\varepsilon}\|_{L^\infty(0,T;L^2)}\leq C\varepsilon^{k}, \quad (5.17)$$

$$\|\nabla W_T^{k,\varepsilon}\|_{L^\infty(0,T;L^2)}\leq C\varepsilon^{k+1}. \quad (5.18)$$

In an analogous manner, we can prove

$$\frac{d}{dt}\|\nabla_\tau W_T^{k,\varepsilon}\|_{L^2}^2+\|\nabla\nabla_\tau W_T^{k,\varepsilon}\|_{L^2}^2\leq C\varepsilon^{2k+2}, \quad (5.19)$$

$$\varepsilon^2\|\nabla\nabla_\tau\vec{W}_u^{k,\varepsilon}\|_{L^2}^2+\|\nabla_\tau\vec{W}_u^{k,\varepsilon}\|_{L^2}^2\leq C\varepsilon^{2k+2}. \quad (5.20)$$

Therefore,

Lemma 5.4

$$\|\nabla_\tau\vec{W}_u^{k,\varepsilon}\|_{L^\infty(0,T;L^2)}\leq C\varepsilon^{k+1},\|\nabla\nabla_\tau\vec{W}_u^{k,\varepsilon}\|_{L^\infty(0,T;L^2)}\leq C\varepsilon^{k}, \quad (5.21)$$

$$\|\nabla\nabla_\tau W_T^{k,\varepsilon}\|_{L^2(0,T;L^2)}\leq C\varepsilon^{k+1}, \quad (5.22)$$

$$\|\nabla\nabla_\tau W_T^{k,\varepsilon}\|_{L^\infty(0,T;L^2)}^2\leq C\varepsilon^{2k+2}+C\varepsilon^{k+1}\|\frac{\partial}{\partial t}\nabla_\tau W_T^{k,\varepsilon}\|_{L^2}. \quad (5.23)$$

For the uniform estimate, we now deduce the $L^\infty(0,T;L^2)$ estimate of $\frac{\partial}{\partial t}\nabla_\tau W_T^{k,\varepsilon}$.

Differentiating Eqs. (5.4) in x and denoting $I_2=\frac{\partial}{\partial x}\vec{W}_u^{k,\varepsilon}$, $I_{12}=\frac{\partial^2}{\partial t\partial x}\vec{W}_u^{k,\varepsilon}$, $J_2=\frac{\partial}{\partial x}W_T^{k,\varepsilon}$, and $J_{12}=\frac{\partial^2}{\partial t\partial x}W_T^{k,\varepsilon}$, we have the following equations

$$-\varepsilon^2\Delta I_{12}+I_{12}+\nabla\frac{\partial^2}{\partial t\partial x}p^{k,\varepsilon}=\gamma J_{12}\vec{k}+\varepsilon^{k+1}\frac{\partial^2}{\partial t\partial x}\tilde{f}^{k,1}+\varepsilon^{k+1}\frac{\partial^2}{\partial t\partial x}\tilde{f}^{k,2}, \quad (5.24)$$

$$\nabla\cdot I_{12}=0, \quad (5.25)$$

$$\begin{aligned}&\frac{\partial}{\partial t}J_{12}+\frac{\partial^2}{\partial t\partial x}\vec{u}^\varepsilon\cdot\nabla W_T^{k,\varepsilon}+\frac{\partial}{\partial t}\vec{u}^\varepsilon\cdot\nabla J_2+\frac{\partial}{\partial x}\vec{u}^\varepsilon\cdot\nabla J_1+\vec{u}^\varepsilon\cdot\nabla J_{12}+I_{12}\cdot\nabla T^\varepsilon\\&+I_1\cdot\nabla\frac{\partial}{\partial x}T^\varepsilon+I_2\cdot\nabla\frac{\partial}{\partial t}T^\varepsilon+\vec{W}_u^{k,\varepsilon}\cdot\nabla\frac{\partial^2}{\partial t\partial x}T^\varepsilon n-I_{12}\cdot\nabla W_T^{k,\varepsilon}-I_1\cdot\nabla J_2\\&-I_2\cdot\nabla J_1-\vec{W}_u^{k,\varepsilon}\cdot\nabla J_{12}-\Delta J_{12}n\\&=\varepsilon^{k+1}\frac{\partial^2}{\partial t\partial x}\tilde{g}^{k,1}+\varepsilon^{k+1}\frac{\partial^2}{\partial t\partial x}\tilde{g}^{k,2},\end{aligned} \quad (5.26)$$

$$I_{12}=0, J_{12}=0 \text{ at } z=0, z=1, \quad (5.27)$$

$$J_{12}=0 \text{ at } t=0. \quad (5.28)$$

Multiplying Eqs. (5.24}) by I_{12} and integrating in Ω, then repeating the same procedure as in the precious proof, we have

$$\varepsilon^2\|\nabla I_{12}\|_{L^2}^2+\| I_{12}\|_{L^2}^2\leq C\| J_{12}\|_{L^2}^2+C\varepsilon^{2k+2}. \quad (5.29)$$

Multiplying Eqs. (5.26) by J_{12} and integrating in Ω, we have

$$\int_\Omega\frac{\partial}{\partial t}J_{12}\cdot J_{12}=\frac{1}{2}\frac{d}{dt}\| J_{12}\|_{L^2}^2, \quad (5.30)$$

$$
\begin{aligned}
&|\int_\Omega \frac{\partial^2}{\partial t\partial x}(\vec{u}^j+\vec{\theta}^j)\cdot\nabla W_T^{k,\varepsilon}\cdot J_{12}| \\
&\le (\|\frac{\partial^2}{\partial t\partial x}\vec{u}^j\|_{L^\infty}+\|\frac{\partial^2}{\partial t\partial x}\vec{\theta}^j\|_{L^\infty})\|\nabla W_T^{k,\varepsilon}\|_{L^2}\|J_{12}\|_{L^2} \\
&\le C\|J_{12}\|_{L^2}^2+C\varepsilon^{2k+2},
\end{aligned}
\tag{5.31}
$$

$$
\begin{aligned}
|\int_\Omega \frac{\partial}{\partial t}(\vec{u}^j+\vec{\theta}^j)\cdot\nabla J_2\cdot J_{12}| &\le (\|\frac{\partial}{\partial t}\vec{u}^j\|_{L^\infty}+\|\frac{\partial}{\partial t}\vec{\theta}^j\|_{L^\infty})\|\nabla J_2\|_{L^2}\|J_{12}\|_{L^2} \\
&\le C\|J_{12}\|_{L^2}^2+C\|\nabla J_2\|_{L^2}^2,
\end{aligned}
\tag{5.32}
$$

$$
\begin{aligned}
|\int_\Omega \frac{\partial}{\partial x}(\vec{u}^j+\vec{\theta}^j)\cdot\nabla J_1\cdot J_{12}| &\le (\|\frac{\partial}{\partial x}\vec{u}^j\|_{L^\infty}+\|\frac{\partial}{\partial x}\vec{\theta}^j\|_{L^\infty})\|\nabla J_1\|_{L^2}\|J_{12}\|_{L^2} \\
&\le C\|J_{12}\|_{L^2}^2+C\|\nabla J_1\|_{L^2}^2,
\end{aligned}
\tag{5.33}
$$

$$
\int_\Omega \vec{u}^\varepsilon\cdot J_{12}\cdot J_{12}=0,
\tag{5.34}
$$

$$
\begin{aligned}
|\int_\Omega I_{12}\cdot\nabla(T^j+\vartheta^j)\cdot J_{12}| &\le (\|\nabla T^j\|_{L^\infty}+\|\nabla\vartheta^j\|_{L^\infty})\|I_{12}\|_{L^2}\|J_{12}\|_{12} \\
&\le C\|I_{12}\|_{L^2}^2+C\|J_{12}\|_{L^2}^2,
\end{aligned}
\tag{5.35}
$$

$$
\begin{aligned}
|\int_\Omega I_1\cdot\nabla\frac{\partial}{\partial x}(T^j+\vartheta^j)\cdot J_{12}| &\le (\|\nabla\frac{\partial}{\partial x}T^j\|_{L^\infty}+\|\nabla\frac{\partial}{\partial x}\vartheta^j\|_{L^\infty})\|I_1\|_{L^2}\|J_{12}\|_{12} \\
&\le C\|I_{12}\|_{L^2}^2+C\varepsilon^{2k+2},
\end{aligned}
\tag{5.36}
$$

$$
\begin{aligned}
|\int_\Omega I_2\cdot\nabla\frac{\partial}{\partial t}(T^j+\vartheta^j)\cdot J_{12}| &\le (\|\nabla\frac{\partial}{\partial t}T^j\|_{L^\infty}+\|\nabla\frac{\partial}{\partial t}\vartheta^j\|_{L^\infty})\|I_2\|_{L^2}\|J_{12}\|_{12} \\
&\le C\|I_{12}\|_{L^2}^2+C\varepsilon^{2k+2},
\end{aligned}
\tag{5.37}
$$

$$
\begin{aligned}
|\int_\Omega \vec{W}_u^{k,\varepsilon}\cdot\nabla\frac{\partial^2}{\partial t\partial x}(T^j+\vartheta^j)\cdot J_{12}| &\le (\|\nabla\frac{\partial^2}{\partial t\partial x}T^j\|_{L^\infty}+\|\nabla\frac{\partial^2}{\partial t\partial x}\vartheta^j\|_{L^\infty})\|\vec{W}_u^{k,\varepsilon}\|_{L^2}\|J_{12}\|_{12} \\
&\le C\|I_{12}\|_{L^2}^2+C\varepsilon^{2k+2},
\end{aligned}
\tag{5.38}
$$

$$
\begin{aligned}
&\int_\Omega I_{12}\cdot W_T^{k,\varepsilon}\cdot J_{12}| \\
&=|\int_\Omega I_{12}\cdot W_T^{k,\varepsilon}\cdot\nabla J_{12}| \\
&\le\|I_{12}\|_{L^4}\|W_T^{k,\varepsilon}\|_{L^4}\|\nabla J_{12}\|_{L^2} \\
&\le\|I_{12}\|_{H^1}\|W_T^{k,\varepsilon}\|_{H^1}\|\nabla J_{12}\|_{L^2}\text{ (by Gagliardo-Nirenberg inequality)}\|_{L^2}^2, \\
&\le\frac{1}{8}\|\nabla J_{12}\|_{L^2}^2+C\varepsilon^{2k}\|J_{12}\|_{L^2}^2\text{ (by 5.29 and Lemma 5.3)} \\
&\le\frac{1}{8}\|\nabla J_{12}\|_{L^2}^2+C\|J_{12}\|_{L^2}^2,
\end{aligned}
\tag{5.39}
$$

$$
\begin{aligned}
&|\int_\Omega I_1\cdot J_2\cdot J_{12}| \\
&=|\int_\Omega I_1\cdot J_2\cdot\nabla J_{12}| \\
&\le\|I_1\|_{L^4}\|J_2\|_{L^4}\|\nabla J_{12}\|_{L^2} \\
&\le\|I_1\|_{H^1}\|J_2\|_{H^1}\|\nabla J_{12}\|_{L^2}\quad\text{(by Gagliardo-Nirenberg inequality)} \\
&\le\frac{1}{8}\|\nabla J_{12}\|_{L^2}^2+C\varepsilon^{2k}(\varepsilon^{2k+2}+\|\nabla J_2\|_{L^2}^2)\quad\text{(by Lemma 5.3 and 5.4)} \\
&\le\frac{1}{8}\|\nabla J_{12}\|_{L^2}^2+C\|\nabla J_2\|_{L^2}^2+C\varepsilon^{2k+2},
\end{aligned}
\tag{5.40}
$$

$$
\begin{aligned}
& |\int_\Omega I_2 \cdot J_1 \cdot J_{12}| \\
=& |\int_\Omega I_2 \cdot J_1 \cdot \nabla J_{12}| \\
\le& \| I_2\|_{L^4} \| J_1\|_{L^4} \| \nabla J_{12}\|_{L^2} \\
\le& \| I_2\|_{H^1} \| J_1\|_{H^1} \| \nabla J_{12}\|_{L^2} \quad \text{(by Gagliardo-Nirenberg inequality)} \\
\le& \frac{1}{8}\| \nabla J_{12}\|_{L^2}^2 + C\varepsilon^{2k}(\varepsilon^{2k+2} + \| \nabla J_1\|_{L^2}^2) \quad \text{(by Lemma 5.3 and 5.4)} \\
\le& \frac{1}{8}\| \nabla J_{12}\|_{L^2}^2 + C\| \nabla J_1\|_{L^2}^2 + C\varepsilon^{2k+2},
\end{aligned} \tag{5.41}
$$

$$\int_\Omega \vec{W}^{k,\varepsilon} \cdot \nabla J_{12} \cdot J_{12} = 0, \tag{5.42}$$

$$-\int_\Omega \Delta J_{12} \cdot J_{12} = \| \nabla J_{12}\|_{L^2}^2, \tag{5.43}$$

$$
\begin{aligned}
& |\varepsilon^{k+1}\int_\Omega (\frac{\partial^2}{\partial t \partial x}\tilde{g}^{k,1} + \frac{\partial^2}{\partial t \partial x}\tilde{g}^{k,1}) \cdot J_{12}| \\
\le& \varepsilon^{k+1}\| z(1-z)\frac{\partial^2}{\partial t \partial x}\tilde{g}^{k,1}\|_{L^2} \| \frac{J_{12}}{z(1-z)}\|_{L^2} + \varepsilon^{k+1}\| \frac{\partial^2}{\partial t \partial x}\tilde{g}^{k,2}\|_{L^2} \| J_{12}\|_{L^2} \\
\le& \frac{1}{8}\| \nabla J_{12}\|_{L^2}^2 + C\| J_{12}\|_{L^2}^2 + C\varepsilon^{2k+2}.
\end{aligned} \tag{5.44}
$$

Combining (5.29)-(5.44),

$$\frac{d}{dt}\| J_{12}\|_{L^2}^2 + \| \nabla J_{12}\|_{L^2} \le C\| J_{12}\|_{L^2}^2 + C\| \nabla J_1\|_{L^2}^2 + C\| \nabla J_2\|_{L^2}^2 + C\varepsilon^{2k+2}. \tag{5.45}$$

Therefore, by Lemma 5.3 and 5.4,

$$\| J_{12}\|_{L^\infty(0,T;L^2)} \le C\varepsilon^{k+1}. \tag{5.46}$$

In an analogous manner, differentiating Eqs. (5.4) in y, we have

Lemma 5.5

$$\| \frac{\partial}{\partial t}\nabla_\tau W_T^{k,\varepsilon}\|_{L^\infty(0,T;L^2)} \le C\varepsilon^{k+1}. \tag{5.47}$$

Combining Lemma 5.4 and 5.5, we have

Lemma 5.6

$$\| \nabla\nabla_\tau W_T^{k,\varepsilon}\|_{L^\infty(0,T;L^2)} \le C\varepsilon^{k+1}. \tag{5.48}$$

Finally, we conclude the proof by applying an appropriate anisotropic embedding theorem (see e.g. in Ref. [12]),

$$
\begin{aligned}
& \| u\|_{L^\infty(0,T;L^\infty)} \\
\le& C[\| \frac{\partial}{\partial z}u\|_{L^\infty(H_\tau^1 \times L_z^2)} (\| u\|_{L^\infty(H_\tau^1 \times L_z^2)}^{1+\delta} + \| u\|_{L^\infty(H_\tau^1 \times L_z^2)}^{\frac{2-\delta}{2}} \| \frac{\partial}{\partial z}u\|_{L^\infty(H_\tau^1 \times L_z^2)}^{\frac{3\delta}{2}})]^{\frac{1}{2+\delta}},
\end{aligned} \tag{5.49}
$$

where $0 < \delta < 2$.

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